Some remarks on the joint estimation of the index and the scale parameter for stable processes.

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SOME REMARKS ON THE JOINT ESTIMATION OF THE INDEX AND THE SCALE PARAMETER FOR STABLE PROCESSES

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ABSTRACT

We deal with joint estimation of index α and scale parameter ξ in a statistical model where one observes all jumps of a stable increasing process with height not less than Y_n and up to time T_n, or more generally all points of a certain Poisson random measure in a window $[0,T_n]\times[Y_n,\infty)$. For different types of asymptotic behaviour of T_n, Y_n as $n\to\infty$, we investigate local asymptotic normality of the model at a true parameter value (α_0,ξ_0), and properties of maximum likelihood estimators at this point.

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1 - INTRODUCTION

Consider a non-decreasing stable process $X=(X_t)_{t\geq 0}$ without drift, with index $\alpha\in(0,1)$ and scale parameter $\xi>0$, that is whose Laplace transform is:

$$\lambda \longrightarrow E(\exp -\lambda X_t) = \exp -t\xi\Gamma(1-\alpha)\lambda^{\alpha}. \tag{1.1}$$

The question of estimating the pair (α,ξ) has been motivated and studied by Basawa and Brockwell [1], [2].

In fact for any $t>0$, the laws of X for different values of the parameters (α,ξ) are all mutually singular, so the inference problem is trivial when one can observe the path of X over any finite interval $[0,t]$. In practice, however, it is usually impossible to actually observe the path of X, while various feasible partial observation schemes are possible. For example Basawa and Brockwell proposed to observe all jumps of X occuring in a fixed time interval $[0,t]$ and of size bigger than y (the laws of X restricted to the corresponding observed σ-field are now all equivalent): as $y\to 0$ they proved the consistency and asymptotic normality of the MLE $(\hat{\alpha},\hat{\xi})$, with the following limiting distribution if the true value is (α,ξ) and $W\in\mathcal{N}(0,1)$:

$$(t\xi y^{-\alpha})^{1/2}\begin{bmatrix}\hat{\alpha}-\alpha\\ (\hat{\xi}-\xi)/\log y\end{bmatrix} \longrightarrow \begin{bmatrix}\alpha W\\ \xi\alpha W\end{bmatrix}. \tag{1.2}$$

They did not investigate LAN (local asymptotic normality), nor did they prove efficiency for the MLE. They also mention that similar results are true for different observation schemes, where t is still fixed and one observes the biggest jumps of X within $[0,t]$ down to the n^{th} one.

In this paper we consider more general observation schemes based on the observation of (some) jumps of X, or equivalently of (part of) the random measure μ associated with the jumps, that is $\mu = \sum_{s>0,\Delta X_s\neq 0}\delta_{(s,\Delta X_s)}$ (δ_a is the Dirac measure sitting at point a). Under (1.1), μ is a Poisson measure with intensity on $\mathbb{R}_+^2$:

$$\alpha\xi\, x^{-(1+\alpha)}\, 1_{(0,\infty)}(t)\, 1_{(0,\infty)}(x)\, dt\, dx \tag{1.3}$$

and conversely if μ is above, then $X_t = \int_0^t\int_0^\infty x\, \mu(ds,dx)$ is a stable process satisfying (1.1). As a matter of fact, we can even consider Poisson measures having intensity measure (1.3) with $\alpha>0$ (corresponding to a stable process only when $\alpha\in(0,1)$), and we will call $(S_i,X_i)_{i\geq 1}$ the (random) points in $\mathbb{R}_+^2$

which support the measure μ. Among many possibilities, we will study three different observation schemes, which can be described as follows in terms of the asymptotic:

Scheme 1: For each n we fix $y_n>0$, $t_n>0$ and we observe all points (S_i,X_i) inside $[0,t_n]\times[y_n,\infty)$. The pair (y_n,t_n) is chosen so that the mean number of observed points is again n, that is t_n and y_n are related by

$$\xi_0 t_n y_n^{-\alpha_0} = n, \tag{1.4}$$

where (ξ_0,α_0) is the true value of the parameter.

Scheme 2: For each n we fix $y_n>0$; we can rearrange the points (S_i,X_i) inside $[0,\infty)\times[y_n,\infty)$ as a sequence (S'_i,X'_i) with $S'_1<S'_2<\ldots$, and we observe the n points $(S'_1,X'_1),\ldots,(S'_n,X'_n)$.

Scheme 3: For each n we fix $t_n>0$; we can rearrange the points (S_i,X_i) inside $[0,t_n]\times(0,\infty)$ as a sequence (S'_i,X'_i) with $X'_1>X'_2>\ldots$, and we observe the n points $(S'_1,X'_1),\ldots,(S'_n,X'_n)$.

The first method proposed by Basawa and Brockwell corresponds to Scheme 1 with $t_n = t$, and thus $y_n = (\xi_0 t_n/n)^{1/\alpha_0}$. The second one corresponds to Scheme 3 with $t_n=t$. Schemes 2 and 3 may look more appealing since the asymptotic does not depend on the true value (α_0,ξ_0) (at least explicitely); while Scheme 1 may be easier to handle practically since it involves the observation over a time-space window which is prescribed before the actual observation.

Now about the results. We denote by $(\hat{\alpha}_n,\hat{\xi}_n)$ the MLE for (α,ξ) at stage n. First, for all observation schemes above, and regardless on the way y_n or t_n behave, $\sqrt{n}(\hat{\alpha}_n-\alpha)/\alpha$ converges in law to $\mathcal{N}(0,1)$ as $n\to\infty$, if α is the true value of the index parameter. Second, if $y_n\to y\in(0,\infty)$ for Schemes 1 and 2, or $t_n/n\to T\in(0,\infty)$ for Scheme 3, we have the LAN property for the model localized around the true value in the usual way, with localizing rate $\sqrt{n}$ for both components, and the MLE is asymptotically regular and efficient for (α,ξ) in the sense of Hájek [4].

Things are more complicated when $y_n\to 0$ or $y_n\to\infty$ for Schemes 1 and 2, or $t_n/n\to 0$ or $t_n/n\to\infty$ for Scheme 3, as is apparent from Basawa and Brockwell's result quoted above. The usual localization leads to a rate $\sqrt{n}$ for the component ξ, and a (faster) rate $\sqrt{n}\ |\log y_n|$ or $\sqrt{n}\ |\log t_n/n|$ for the component α, and the LAN property holds, but the limiting Gaussian model is degenerate (the corresponding Fisher information matrix is non-invertible). There is also a "non-classical" localization (which is a non-linear transform) giving the LAN property with a non-degenerate limit and thus allowing to check efficiency of estimators: e.g. for Schemes 1 and 2 it is possible to estimate efficiently as $n\to\infty$ functionals $(\alpha,\xi y_n^{-\alpha})$ of (α,ξ), which is relevant since then the observation describes the trajectory of a Markov step process $\int_0^{\cdot}\int_{y_n}^{\infty} x\, \mu(ds,dx)$ where exponential holding times in any state have parameter $\xi y_n^{-\alpha}$ (cf. Theorem 1 and Remark 4).

Finally we investigate one-dimensional submodels where ξ is a known function of α. The situation is then the opposite of the previous one: if $y_n\to y\in(0,\infty)$ or $t_n/n\to T\in(0,\infty)$, it may happen (depending on the function above) that the MLE is not consistent. If $y_n\to 0$ or $y_n\to\infty$ (resp. $t_n/n\to 0$ or $t_n/n\to\infty$), there is a localization with the LAN property, and the MLE is efficient, and the rate is $\sqrt{n}\ |\log y_n|$ or $\sqrt{n}\ |\log t_n/n|$, which is better than what is obtained for the complete model. It should be noted at this point that in this case the rate of convergence depends on the observation schemes: it is better to have $|\log y_n|$ or $|\log t_n/n|$ be as big as possible, that is we should observe the points either on a very small time interval and down to a very small value of the "size", or the other way around points with a very big size only, on a very large time interval.

2 - THE MODEL

1) Let $(\Omega,\mathcal{F})$ be the canonical space of all σ-finite integer-valued measures on $\mathbb{R}_+^2$, with the canonical measure $\mu = \sum_{i\geq 1} \delta_{(S_i,X_i)}$. For all $\alpha>0$, $\xi>0$ we call $P_{\alpha,\xi}$ the unique probability measure on $(\Omega,\mathcal{F})$ under which μ is a Poisson random measure with intensity given by (1.3). We call $\mathcal{F}_t^y$ the σ-field generated by the restriction of μ to $[y,\infty)\times[0,t]$. If $y>0$ and $t<\infty$, the restrictions of the measures $P_{\alpha,\xi}$ to $\mathcal{F}_t^y$ are all equivalent, and the density of the restriction of $P_{\alpha',\xi'}$ w.r.t. the restriction of $P_{\alpha,\xi}$ is

$$Z_t^y(\alpha',\xi'|\alpha,\xi) = \exp\{t(\xi y^{-\alpha} - \xi' y^{-\alpha'}) + N_t^y \log(\frac{\xi'\alpha'}{\xi\alpha}) + \overline{N}_t^y(\alpha-\alpha')\}, \qquad (2.1)$$

where

$$N_t^y = \sum_{i\geq 1} 1_{\{X_i\geq y, S_i\leq t\}}, \qquad \overline{N}_t^y = \sum_{i\geq 1} 1_{\{X_i\geq y, S_i\leq t\}} \log(X_i). \qquad (2.2)$$

In the other cases ($y=0$ or $t=\infty$ or both) the restrictions of the measures

$P_{\alpha,\xi}$ to $\mathcal{F}^y_t$ are all mutually singular.

Formula (2.1) extends to stopping times. Namely if T is a finite stopping time for the filtration $(\mathcal{F}^y_t)_{t\geq 0}$ then $Z^y_T(.)$ given by (2.1) with t substituted with T is the relative density of the absolutely continuous part of $P_{\alpha',\xi'}$ w.r.t. $P_{\alpha,\xi}$ in restriction to $\mathcal{F}^y_T$. Similarly $(\mathcal{F}^{1/z}_t)_{z\geq 0}$ is a filtration and if $1/Y$ is a finite stopping time relative to this filtration, the density above in restriction to $\mathcal{F}^Y_t$ is $Z^Y_t(.)$ given by (2.1) with y substituted with Y.

2) In what follows the true value of the parameter is (α_0,ξ_0). Now we consider the observation schemes described in the introduction, and introduce some unified notation:

Scheme 1: At stage n we have y_n, t_n related by (1.4), and we observe $\mathcal{F}^{y_n}_{t_n}$. We set $Y_n=y_n$, $T_n=t_n$.

Scheme 2: At stage n we have y_n, and we observe $\mathcal{F}^{y_n}_{T_n}$ where $T_n = \inf(t: N^{y_n}_t=n)$ is an $(\mathcal{F}^{y_n}_t)_{t\geq 0}$-stopping time. Set $Y_n=y_n$.

Scheme 3: At stage n we have t_n, and we observe $\mathcal{F}^{Y_n}_{t_n}$, where $Y_n = \sup(y: N^y_{t_n}=n)$, so that $1/Y_n$ is an $(\mathcal{F}^{1/z}_{t_n})_{z\geq 0}$-stopping time. Set $T_n=t_n$.

In all cases, we set

$$N_n = N^{Y_n}_{T_n}, \qquad \overline{N}_n = \overline{N}^{Y_n}_{T_n}, \qquad L_n = \log Y_n. \tag{2.3}$$

These are observable, and $N_n=n$ for schemes 2 and 3, and L_n is deterministic for schemes 1 and 2. The density of $P_{\alpha,\xi}$ w.r.t. P_{α_0,ξ_0}, in restriction to the observed σ-field $\mathcal{F}^{Y_n}_{T_n}$, is

$$Z_n(\alpha,\xi) = \exp\{T_n(\xi_0 e^{-\alpha_0 L_n} - \xi e^{-\alpha L_n}) + N_n \log\frac{\alpha\xi}{\alpha_0\xi_0} - \overline{N}_n(\alpha-\alpha_0)\}. \tag{2.4}$$

By a simple computation, the MLE for the pair (α,ξ) at stage n is

$$\hat{\alpha}_n = N^n/(\overline{N}_n - N_n L_n), \qquad \hat{\xi}_n = (N_n/T_n)\exp \hat{\alpha}_n L_n. \tag{2.5}$$

REMARK 1: When $\alpha<1$, so $X_t = \int_0^t\int_0^\infty x\ \mu(ds,dx)$ is under $P_{\alpha,\xi}$ a stable process with (1.1), then $P_{\alpha,\xi}(X_1>r) \sim \xi r^{-\alpha}$ as $r\to\infty$ (see e.g. Feller [3]). In this case one can view $\hat{\alpha}_n$ and $\hat{\xi}_n$ as continuous-time versions of the estimators introduced by Hill [6] for the index of regular variation α and the tickness ξ of the tail of the law of X_1: see also Hall [5]. ■

3) In fact, the asymptotic behaviour of the following variables plays an essential rôle:

$$U_n = \frac{1}{\sqrt{n}}[N_n + \alpha_0(N_nL_n - \overline{N}_n)], \qquad V_n = (N_n-n)/\sqrt{n}, \qquad W_n = (T_n\xi_0e^{-\alpha_0L_n} - n)/\sqrt{n}.$$

LEMMA 1: a) *For scheme 1 we have* $W_n = 0$ *and* (U_n,V_n) *converges in law under* P_{α_0,ξ_0} *to* $\mathcal{N}(0,I_2)$.

b) *For schemes 2 and 3 we have* $V_n = 0$ *and* (U_n,W_n) *converges in law under* P_{α_0,ξ_0} *to* $\mathcal{N}(0,I_2)$.

Proof. Throughout the proof, $P=P_{\alpha_0,\xi_0}$. For Scheme 1 (1.4) yields $W_n=0$, while in Schemes 2 and 3 we have $N_n=n$, hence $V_n=0$.

a) For Scheme 1, a standard computation on Poisson measures yields

$$\varphi_n(u,v) \ := \ \log E[\exp(-uN_n-v\overline{N}_n)] \ = \ -n + n\alpha_0e^{-u-vL_n}\frac{1}{\alpha_0+v}.$$

Therefore (here L_n is deterministic):

$$\log E[e^{-uU_n-vV_n}] \ = \ v\sqrt{n} + \varphi_n[(u+u\alpha_0L_n+v)/\sqrt{n},-\alpha_0u/\sqrt{n}]$$

$$= \ -n + v\sqrt{n} + ne^{-(u+v)/\sqrt{n}}\frac{1}{1-u/\sqrt{n}} \ \longrightarrow \ \frac{1}{2}(u^2 + v^2)$$

and the claim follows.

b1) Consider now Scheme 2. The points (S_i,X_i) inside $[0,\infty)\times[y_n,\infty)$ can be labelled as (S'_i,X'_i) with $0<S'_1<S'_2<\ldots$, and both sequences $(S'_i)_{i\geq1}$ and $(X'_i)_{i\geq1}$ are independent, and the second one consists in i.i.d. variables with density $x \to \alpha_0x^{-1-\alpha_0}y_n^{\alpha_0}1_{(y_n,\infty)}(x)$, and the first one is a Poisson process on $\mathbb{R}_+$ with intensity $\xi_0y_n^{-\alpha_0}$, and further $T_n = S'_n$. Recalling that $N_n=n$ and

$Y_n=y_n$ we observe that U_n and W_n are independent.

We have $\overline{N}_n = \sum_{1\le i\le n} \log X'_i$ and $E(\exp -u \log X'_i) = e^{-uL_n} \frac{\alpha_0}{\alpha_0+u}$, hence

$$\log E(e^{uU_n}) = u\sqrt{n} - n\log(1+u/\sqrt{n}) \longrightarrow u^2/2 \qquad (2.6)$$

and U_n converges in law to $\mathcal{N}(0,1)$. Next T_n has a gamma distribution with index n and parameter $\xi_0 e^{-\alpha_0 L_n}$, so

$$\log E(e^{-uW_n}) = u\sqrt{n} - n\log(1+u/\sqrt{n}) \longrightarrow u^2/2 \qquad (2.7)$$

and W_n converges in law to $\mathcal{N}(0,1)$. Then the claim is proved.

b2) Finally consider Scheme 3. The points (S_i,X_i) inside $[0,t_n]\times(0,\infty)$ can be labelled as (S'_i,X'_i) with $X'_1>X'_2>\ldots$, and if $Z_i = X_i'^{-\alpha_0}$ the sequence $(Z_i)_{i\ge1}$ is a Poisson process on $\mathbb{R}_+$ with intensity $\xi_0 t_n$. Further, we have $\overline{N}_n = -\frac{1}{\alpha_0}\sum_{1\le i\le n} \log Z_i$, and $U_n = \sqrt{n} + \frac{1}{\sqrt{n}}\sum_{1\le i\le n} \log(Z_i/Z_n)$, and $e^{-\alpha_0 L_n} = Z_n$. First Z_n has a gamma distribution with index n and parameter $\xi_0 t_n$, hence (2.7) holds. Second the family $(Z_i/Z_n : 1\le i\le n-1)$ is independent from Z_n and has the same law as the increasing rearrangment of $n-1$ independent variables, uniformly distributed over $(0,1)$. So U_n and W_n are independent and a simple computation shows that (2.6) holds: our claim is thus proved. ∎

Note that (2.5) yields

$$\sqrt{n}\,\frac{\hat{\alpha}_n - \alpha_0}{\alpha_0} = \frac{U_n}{1 + (V_n - U_n)/\sqrt{n}} \qquad (2.8)$$

and thus, as a first consequence of Lemma 1 we readily obtain

COROLLARY 1: *For all observation schemes,* $\sqrt{n}\,\frac{\hat{\alpha}_n - \alpha_0}{\alpha_0}$ *converges in law under* P_{α_0,ξ_0} *to* $\mathcal{N}(0,1)$*, where* $\hat{\alpha}_n$ *is the first component of the MLE estimator.*

This is a natural extension of the asymptotic behaviour of the Hill estimator, as expressed e.g. in Theorem 2 of Hall [5]. Lemma 1 also contains information about the second component $\hat{\xi}_n$ of the MLE. Let us first introduce the following *deterministic* sequence

$$\ell_n = \alpha_0 L_n \quad \text{for Scheme 1 and 2,} \qquad \ell_n = \log t_n \xi_0 / n \quad \text{for Scheme 3.} \tag{2.9}$$

Since $\ell_n - \alpha_0 L_n = \log(1+W_n/\sqrt{n})$ for Scheme 3, we have in all cases by Lemma 1:

$$\ell_n \longrightarrow \ell \in [-\infty,+\infty] \Leftrightarrow L_n \xrightarrow{P_{\alpha_0,\xi_0}} \ell/\alpha_0, \tag{2.10}$$

$$\ell_n/\sqrt{n} \longrightarrow 0 \Leftrightarrow L_n/\sqrt{n} \xrightarrow{P_{\alpha_0,\xi_0}} 0 \Leftrightarrow \begin{cases} (\log y_n)/\sqrt{n} \to 0 & \text{(Schemes 1,2)} \\ (\log t_n)/\sqrt{n} \to 0 & \text{(Scheme 3)} \end{cases} \tag{2.11}$$

COROLLARY 2: a) *For all observation schemes, the sequence* $(\hat{\xi}_n)$ *is weakly consistent at* ξ_0 *if and only if (2.11) holds.*

b) *Condition (2.11) together with* $|\ell_n| \longrightarrow \infty$ *yield*

$$\frac{\sqrt{n}}{\xi_0 \ell_n}(\hat{\xi}_n - \xi_0) = \frac{\sqrt{n}}{\alpha_0}(\hat{\alpha}_n - \alpha_0) + o_{P_{\alpha_0,\xi_0}}(1). \tag{2.12}$$

c) *Condition (2.10) with* $\ell \in (-\infty,+\infty)$ *implies*

$$\left(\frac{\sqrt{n}}{\alpha_0}(\hat{\alpha}_n - \alpha_0), \frac{\sqrt{n}}{\xi_0}(\hat{\xi}_n - \xi_0)\right) = (U_n, \ell U_n + V_n - W_n) + o_{P_{\alpha_0,\xi_0}}(1),$$

which converges in law under P_{α_0,ξ_0} *to a centered Gaussian variable with co-variance matrix* $\begin{bmatrix} 1 & \ell \\ \ell & 1+\ell^2 \end{bmatrix}$.

(b) and Corollary 1 give an extension of Basawa and Brockwell's result.

Proof. (2.5) implies

$$\frac{\hat{\xi}_n}{\xi_0} = \frac{1+V_n/\sqrt{n}}{1+W_n/\sqrt{n}} \exp\left[\frac{\sqrt{n}}{\alpha_0}(\hat{\alpha}_n - \alpha_0)\frac{L_n \alpha_0}{\sqrt{n}}\right]. \tag{2.13}$$

Thus Lemma 1 and Corollary 1 give all claims. ∎

3 - LOCALIZATION

1) In order to prove that in some of the schemes discussed above the MLE is asymptotically optimal, we need to localize the statistical model around the true value (α_0,ξ_0). The first localization which we propose is not traditional, in the sense that it is not an affine transformation of the initial coordinates. It gives a sequence of "curved" statistical models, the curve actually depending on the value of n.

In schemes 1 and 2, $L_n = \log y_n$ is deterministic and the parameters (γ,η) of the local model around (α_0,ξ_0) at stage n are given by

$$\alpha = \alpha_0(1+\gamma/\sqrt{n}), \qquad \xi = \xi_0(1+\eta/\sqrt{n})e^{L_n\gamma\alpha_0/\sqrt{n}}. \tag{3.1}$$

For Scheme 3 this is not feasible since L_n is random, and we set instead:

$$\alpha = \alpha_0(1+\gamma/\sqrt{n}), \qquad \xi = \xi_0(1+\eta/\sqrt{n})\left(\frac{t_n\xi_0}{n}\right)^{\gamma/\sqrt{n}}. \tag{3.2}$$

REMARK 2: (3.1) and (3.2) coincide for Scheme 1, because of (1.4). ∎

REMARK 3: At first glance this localization may seem a bit strange. To get some insight, let us consider only Schemes 1 and 2 and reparametrize the model by (α,β_n), with

$$\beta_n = \beta_n(\alpha,\xi) := \xi y_n^{-\alpha} \qquad (\Leftrightarrow\ \xi = \beta_n y_n^{\alpha}).$$

This reparametrization depending on n is natural for the problem at hand, since at stage n one observes in fact the process $X_t^n = \int_0^t\int_{y_n}^{\infty} x\ \mu(ds,dx)$ on the time interval $[0,t_n]$, and under $P_{\alpha,\xi}$ this process is a step Markov process with parameter $\beta_n(\alpha,\xi)$ for the holding times. The true value of the parameter β_n is $\beta_{n,0} = \xi_0 y_n^{-\alpha_0} = \xi_0 e^{-\alpha_0 L_n}$. The local parameters (γ,η) at stage n are given in function of (α,β_n) by

$$\alpha = \alpha_0(1+\gamma/\sqrt{n}), \qquad \beta_n = \beta_{n,0}(1+\eta/\sqrt{n}). \tag{3.3}$$

So (3.1) yields a local model in the usual sense for the (α,β_n)-reparametrization. The MLE $\hat{\beta}_n = N_n/T_n$ for β_n is always consistent at (α_0,ξ_0), i.e. $\hat{\beta}_n/\beta_{n,0} \longrightarrow 1$ in P_{α_0,ξ_0}-probability, because $\hat{\beta}_n/\beta_{n,0} = (1+V_n/\sqrt{n})(1+W_n/\sqrt{n})$.

(A similar argument leading to (3.3) for Scheme 3 would require $\beta_n = \xi(t_n\xi_0/n)^{-\alpha/\alpha_0}$: here (α_0,ξ_0) appears in the reparametrization, which does not make sense). ∎

At stage n, the space of local parameters is $\Theta_n = (-\sqrt{n},\infty)\times(-\sqrt{n},\infty)$. Note that $\Theta_n\uparrow\mathbb{R}^2$ as $n\uparrow\infty$. For $n\in\mathbb{N}^*$, we set $P^n_{\gamma,\eta} = P_{\alpha,\xi}$ for (γ,η) related with (α,ξ) by (3.1) or (3.2), and $\mathcal{E}_n = (\Omega,\mathcal{F}^{Y}_{T_n},(P^n_{\gamma,\eta})_{(\gamma,\eta)\in\Theta_n})$. Using (2.5) and (2.8), we see that the MLE in the model $\mathcal{E}_n$ is

$$\hat{\gamma}_n = \frac{U_n}{1+(V_n-U_n)/\sqrt{n}}, \qquad \hat{\eta}_n = \begin{cases} \dfrac{V_n - W_n}{1 + W_n/\sqrt{n}} & \text{for Schemes 1,2} \\ \sqrt{n}\,[\,(1+W_n/\sqrt{n})^{-1-\hat{\gamma}_n/\sqrt{n}} - 1] & \text{for Scheme 3.} \end{cases} \tag{3.4}$$

The limiting model will be the Gaussian shift $\mathcal{E} = (\mathbb{R}^2,\mathcal{R}^2,(Q_{\gamma,\eta})_{(\gamma,\eta)\in\mathbb{R}^2})$ where $Q_{\gamma,\eta}$ is the normal distribution on $\mathbb{R}^2$ with covariance I_2 and mean value (γ,η).

THEOREM 1: *For all observation schemes, and as* $n\to\infty$, *we have:*

a) *The sequence* $\mathcal{E}_n$ *weakly converges to* $\mathcal{E}$.

b) *The pair* $(\hat{\gamma}_n,\hat{\eta}_n)$ *converges in law to* $\mathcal{N}(0,I_2)$, *under* P_{α_0,ξ_0}.

c) *The sequence* $(\hat{\gamma}_n,\hat{\eta}_n)$ *is a central sequence for* $\mathcal{E}_n$.

Proof. Throughout the proof, $P=P_{\alpha_0,\xi_0}$. Set $\overline{Z}_n(\gamma,\eta) = Z_n(\alpha,\xi)$ with (α,ξ) and (γ,η) related by (3.1) or (3.2). For schemes 1 and 2 (resp. 3) we have

$$\log \overline{Z}_n(\gamma,\eta) = -\eta T_n\xi_0 e^{-\alpha_0 L_n}/\sqrt{n} + N_n[\log(1+\eta/\sqrt{n})(1+\gamma/\sqrt{n}) + \alpha_0 L_n\gamma/\sqrt{n}] - \overline{N}_n\alpha_0\gamma/\sqrt{n}$$

$$(\text{resp. } T_n\xi_0 e^{-\alpha_0 L_n}[1 - (1+\eta/\sqrt{n})(T_n\xi_0 e^{-\alpha_0 L_n}/n)^{\gamma/\sqrt{n}}]$$

$$+ N_n[\log(1+\eta/\sqrt{n})(1+\gamma/\sqrt{n}) + (\log T_n\xi_0/n)\gamma/\sqrt{n}] - \overline{N}_n\alpha_0\gamma/\sqrt{n}\ ).$$

For Scheme 1, we get

$$\log \overline{Z}_n(\gamma,\eta) = -\eta\sqrt{n} + \gamma U_n + (n + V_n\sqrt{n})[\log(1+\eta/\sqrt{n})(1+\gamma/\sqrt{n}) - \gamma/\sqrt{n}]$$

$$= \gamma U_n + \eta V_n - (\gamma^2+\eta^2)/2 + o_P(1). \tag{3.5}$$

For Scheme 2, we get

$$\log \overline{Z}_n(\gamma,\eta) = -\eta\sqrt{n} - \eta W_n + \gamma U_n + n[\log(1+\eta/\sqrt{n})(1+\gamma/\sqrt{n}) - \gamma/\sqrt{n}]$$

$$= \gamma U_n - \eta W_n - (\gamma^2+\eta^2)/2 + o_P(1). \tag{3.6}$$

For Scheme 3 we get, with $W'_n = 1+W_n/\sqrt{n}$ and $W''_n = \sqrt{n}\log W'_n$:

$$\log \overline{Z}_n(\gamma,\eta) = nW'_n[1 - (1+\eta/\sqrt{n})W'^{\gamma/\sqrt{n}}_n] + \gamma W''_n + \gamma U_n + n[\log(1+\eta/\sqrt{n})(1+\gamma/\sqrt{n}) - \gamma/\sqrt{n}]$$

$$= \gamma U_n - \eta W_n - (\gamma^2+\eta^2)/2 + o_P(1). \tag{3.7}$$

(3.5), (3.6) and (3.7) and Lemma 1 prove (a), and also that a central sequence is (U_n,V_n) for Scheme 1 and $(U_n,-W_n)$ for Schemes 2 and 3. (3.4) gives $\hat{\alpha}_n = U_n + o_P(1)$, and also $\hat{\xi}_n = V_n + o_P(1)$ for Scheme 1, and $\hat{\xi}_n = -W_n + o_P(1)$ for Scheme 2 or 3: this proves (b) and (c). ∎

REMARK 4: a) We refer to LeCam [7] or Strasser [8] for the notion of central sequences. As a consequence of this theorem, $(\hat{\gamma}_n,\hat{\eta}_n)$ is an efficient sequence of estimators for the local parameters (γ,η).

b) For Schemes 1 or 2, by Remark 3 and in particular by (3.3), the estimation sequence $(\hat{\alpha}_n,\hat{\beta}_n)$ for (α,β_n) is regular in the sense of Hájek [4] and efficient at $(\alpha_0,\beta_{n,0})$ (i.e. under P_{α_0,ξ_0}): take any other estimating sequence $(\tilde{\alpha}_n,\tilde{\beta}_n)$ ($\mathcal{F}^{y_n}_{T_n}$-measurable), with the property that

$$\mathcal{L}\Big(\frac{\sqrt{n}}{\alpha_0}(\tilde{\alpha}_n-\alpha_0(1+\gamma/\sqrt{n})),\frac{\sqrt{n}}{\beta_{n,0}}(\tilde{\beta}_n-\beta_{n,0}(1+\eta/\sqrt{n}))\Big|P_{\alpha_0(1+\gamma/\sqrt{n}),\beta_{n,0}(1+\eta/\sqrt{n})}\Big) \tag{3.8}$$

converges as $n\to\infty$ to a limit which does not depend on (γ,η): then this limiting law is necessarily more spread out than the limit law $\mathcal{N}(0,I_2)$ obtained for (3.8) with $(\hat{\alpha}_n,\hat{\beta}_n)$ in place of $(\tilde{\alpha}_n,\tilde{\beta}_n)$.

c) However, for Schemes 1 or 2, no conclusion can be drawn on optimality of $\hat{\xi}_n$ as an estimator of ξ. By Corollary 2, $\hat{\xi}_n$ need not even converge to ξ. ∎

2) Suppose now that $\ell_n \to \ell$ with $\ell\in\mathbb{R}$ (recall (2.9)). Then the second parts of both (3.1) and (3.2) give $\xi \sim \xi_0(1 + \eta/\sqrt{n} + \gamma\ell/\sqrt{n})$ and we are led to con-

sider new local coordinates (γ, H) with

$$\alpha = \alpha_0(1 + \gamma/\sqrt{n}), \qquad \xi = \xi_0(1 + H/\sqrt{n}). \tag{3.9}$$

We call $\mathcal{E}'_n$ the local model at stage n, with local parameters (3.9) (it has the same set of parameters Θ_n as above), and $(\hat{\gamma}_n, \hat{H}_n)$ the corresponding MLE. The limiting model is the Gaussian model $\mathcal{E}' = (\mathbb{R}^2, \mathcal{R}^2, (Q^{\ell}_{\gamma,H})_{(\gamma,H)\in\mathbb{R}^2})$, where $Q^{\ell}_{\gamma,H}$ is the Gaussian distribution with mean (η, H) and covariance $I(\ell) = \begin{bmatrix} 1+\ell^2 & -\ell \\ -\ell & 1 \end{bmatrix}$.

Since H and $\eta+\gamma\ell$ are "equivalent", an affine transformation readily gives the following corollary (which can of course be proved directly):

COROLLARY 3: *When* $\ell_n \to \ell\in\mathbb{R}$ *we have as* $n\to\infty$:

a) *The sequence* $\mathcal{E}'_n$ *weakly converges to* $\mathcal{E}'$.

b) *The pair* $(\hat{\gamma}_n, \hat{H}_n)$ *converges in law to* $\mathcal{N}(0, I(\ell)^{-1})$, *under* P_{α_0,ξ_0}.

c) *The sequence* $(\hat{\gamma}_n, \hat{H}_n)$ *is a central sequence for* $\mathcal{E}'_n$.

REMARK 5: a) Part (b) above is just a reformulation of Corollary 2(c).

b) Now the pair $(\hat{\alpha}_n, \hat{\xi}_n)$ is regular and efficient for (α,ξ) at (α_0,ξ_0). ∎

3) There is also a third localization which gives the LAN property. This third localization being equivalent to (3.9) under the assumptions of Corollary 3, we consider only the case $|\ell_n| \to \infty$. This localization is motivated by the correct choice of norming constants, taken *separately*, for the α- and ξ-components of the score function at (α_0,ξ_0). Set

$$\alpha = \alpha_0(1 + \Gamma/\ell_n\sqrt{n}), \qquad \xi = \xi_0(1 + H/\sqrt{n}). \tag{3.10}$$

(for all n big enough $\ell_n \neq 0$, so γ is well defined. We call $\mathcal{E}''_n$ the local model at stage n, with local parameters (3.10), and $(\hat{\Gamma}_n, \hat{H}_n)$ the corresponding MLE. The limiting model is the Gaussian model $\mathcal{E}'' = (\mathbb{R}^2, \mathcal{R}^2, (R_{\gamma,H})_{(\gamma,H)\in\mathbb{R}^2})$, where $R_{\gamma,H}$ is the Gaussian distribution with mean (η, H) and covariance $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

THEOREM 2: *If* $|\ell_n| \to \infty$, *the sequence* $\mathcal{E}''_n$ *weakly converges to* $\mathcal{E}''$ *as* $n\to\infty$.

The model $\mathcal{E}''$ is degenerate, with non-invertible Fisher information matrix. The MLE $(\hat{\Gamma}_n,\hat{H}_n)$ do not converge in law here. This is of course in accordance with Corollary 2.

Proof. Set $\bar{Z}''_n(\Gamma,H) = Z_n(\alpha,\xi)$ with (Γ,H) and (α,ξ) related by (3.9). Then

$$\log \bar{Z}''_n(\Gamma,H) = e^{-\alpha_0 L_n} T_n \xi_0 [1 - (1+H/\sqrt{n}) e^{-\alpha_0 \Gamma L_n/\ell_n\sqrt{n}}] + \Gamma U_n/\ell_n$$
$$+ (n+V_n\sqrt{n})[\log(1+H/\sqrt{n})(1+\Gamma/\ell_n\sqrt{n}) - \Gamma(1+\alpha_0 L_n)/\ell_n\sqrt{n}]$$

Recall that for Schemes 1 and 2 we have $\ell_n = \alpha_0 L_n$, hence

$$\log \bar{Z}''_n(\Gamma,H) = (n+W_n\sqrt{n})[1 - (1+H/\sqrt{n})e^{-\Gamma/\sqrt{n}}] + \Gamma U_n/\ell_n$$
$$+ (n+V_n\sqrt{n})[\log(1+H/\sqrt{n})(1+\Gamma/\ell_n\sqrt{n}) - \Gamma/\ell_n\sqrt{n} - \Gamma/\sqrt{n}]$$
$$= -(\Gamma-H)^2/2 + o_P(1) + \begin{cases} V_n(H-\Gamma) & \text{for Scheme 1} \\ W_n(\Gamma-H) & \text{for Scheme 2,} \end{cases}$$

and the result is proved in these two cases. For Scheme 3 we have $\alpha_0 L_n = \ell_n - W''_n/\sqrt{n}$ (notation of the proof of Theorem 1), so another similar computation shows that $\log \bar{Z}''_n(\Gamma,H) = W_n(\Gamma-H) - (\Gamma-H)^2/2 + o_P(1)$, and we are finished. ∎

4 – ONE-DIMENSIONAL SUBMODELS

For $A\subseteq(0,\infty)$ open and $f\colon A \to (0,\infty)$ twice continuously differentiable, we look at submodels $\Theta_f = \{(\alpha,f(\alpha))\colon \alpha\in A\}\subseteq\Theta$, under the same observation schemes as before. Of course we write $P_\alpha = P_{\alpha,f(\alpha)}$. An example is given by $A=(0,1)$ and $f(\alpha) = 1/\Gamma(1-\alpha)$: this corresponds to the observation of an increasing stable process X with Laplace transform $E(\exp -\lambda X_t) = \exp -t\lambda^\alpha$ (cf. (1.1)). Another example is when $\xi=\xi_0$ is known, so $f(\alpha)=\xi_0$ for all $\alpha\in A$.

The true value of α is $\alpha_0\in A$, and $\xi_0=f(\alpha_0)$. We use the notation ℓ_n of (2.9). The localization for α will be the same as in (3.10), except that here we do not assume $|\ell_n|\to\infty$, and so ℓ_n may vanish. Therefore we set, at stage n:

$$\alpha = \alpha_0(1 + \omega_n\Gamma) \qquad \text{with } \omega_n = \omega_n(\alpha_0) := [n(1+\ell_n^2)]^{-1/2}. \tag{4.1}$$

At stage n, the space of local parameters is $\bar{\Theta}_n = \{\Gamma\colon \alpha_0(1+\omega_n\Gamma)\in A\}$, so that

$\overline{\Theta}_n \uparrow \mathbb{R}$ as $n \to \infty$. For $n \in \mathbb{N}^*$, we set $\overline{P}^n_\Gamma = P_\alpha$ for Γ related with α by (4.1), and $\overline{\mathcal{E}}_n = (\Omega, \mathcal{F}^{Y^n}_{T_n}, (\overline{P}^n_\Gamma)_{\Gamma \in \overline{\Theta}_n})$. We consider a MLE $\check{\alpha}_n$ for α in the model $\overline{\mathcal{E}}_n$ (which is different from $\hat{\alpha}_n$ in (2.5)!). By convention we take $\check{\alpha}_n = \alpha_1$ with α_1 arbitrary in A if the likelihood does not reach its maximum inside A, and the corresponding MLE for the local model:

$$\check{\Gamma}_n = (\check{\alpha}_n - \alpha_0)/\alpha_0 \omega_n \tag{4.2}$$

THEOREM 3: a) *If* $\ell_n \to \ell \in \mathbb{R}$, *the sequence* $\overline{\mathcal{E}}_n$ *weakly converges to the model* $(\mathbb{R}, \mathcal{R}, \mathcal{N}(\Gamma, \varphi)_{\Gamma \in \mathbb{R}})$, *where*

$$c = \alpha_0 \frac{f'(\alpha_0)}{f(\alpha_0)}, \qquad \varphi = [1+(c-\ell)^2]/(1+\ell^2). \tag{4.3}$$

b) *If* $|\ell_n| \to +\infty$, *the sequence* $\overline{\mathcal{E}}_n$ *weakly converges to the model* $(\mathbb{R}, \mathcal{R}, \mathcal{N}(\Gamma, 1)_{\Gamma \in \mathbb{R}})$, *and* $\check{\Gamma}_n$ *is a central sequence for* $\overline{\mathcal{E}}_n$ *and converges in law under* P_{α_0} *to* $\mathcal{N}(0,1)$.

Proof. 1) We prove first the convergence of the models in (a) and (b). We use all notation of the proof of Theorem 1, and $P = P_{\alpha_0}$. Note that

$$\log Z_n(\alpha, f(\alpha)) = nW'_n[1 - \frac{f(\alpha)}{f(\alpha_0)} e^{-(\alpha-\alpha_0)L_n}] + N_n \log \frac{f(\alpha)\alpha}{f(\alpha_0)\alpha_0} + \overline{N}_n(\alpha_0 - \alpha). \tag{4.4}$$

Set $\overline{Z}_n(\Gamma) = Z_n(\alpha, f(\alpha))$ with α and Γ related by (4.1). Then

$$\log \overline{Z}_n(\Gamma) = nW'_n[1 - \frac{f(\alpha)}{f(\alpha_0)} e^{-\alpha_0 \omega_n L_n \Gamma}] + \Gamma \omega_n U_n \sqrt{n} + (n+V_n\sqrt{n})[\log \frac{f(\alpha)\alpha}{f(\alpha_0)\alpha_0} - (1+\alpha_0 L_n)\omega_n \Gamma].$$

For Schemes 1 and 2 we have $\alpha_0 L_n = \ell_n$, while $\alpha_0 L_n = \ell_n - W''_n/\sqrt{n}$ for Scheme 3. In all cases $\omega_n L_n \sqrt{n}$ remains bounded in probability, so a simple computation yields with c as in (4.3):

$$\log \overline{Z}_n(\Gamma) = \Gamma \omega_n \sqrt{n}[U_n + (V_n - W_n)(c - \alpha_0 L_n)] - \Gamma^2 \omega_n^2 n[1+(c-\alpha_0 L_n)^2]/2 + o_P(1).$$

Now (a) is obvious, since then $\omega_n\sqrt{n} \to (1+\ell^2)^{-1/2}$ and $\alpha_0 L_n \to \ell$ in probability (at least). In the situation of (b), $\omega_n\sqrt{n} \to 0$ and $|\alpha_0 L_n \omega_n \sqrt{n}| \to 1$, hence if $\overline{V}_n$ and $\overline{W}_n$ equal V_n and W_n multiplied by the sign of ℓ_n, we deduce that $\log \overline{Z}_n(\Gamma) = -\Gamma(\overline{V}_n - \overline{W}_n) - \Gamma^2/2 + o_P(1)$: this proves the convergence of $\overline{\mathcal{E}}_n$ and also that $-\overline{V}_n$ for Scheme 1 and $\overline{W}_n$ for Schemes 2 or 3 is a central sequence. So for (b) it remains to prove that

$$\check{\Gamma}_n = -\overline{V}_n + o_P(1) \quad \text{(Scheme 1)}, \qquad \check{\Gamma}_n = \overline{W}_n + o_P(1) \quad \text{(Schemes 2, 3)}. \tag{4.5}$$

2) Now we assume $|\ell_n| \to \infty$. In a first step we prove that

$$P(|\ell_n(\check{\alpha}_n - \alpha_0)| \le \delta)) \longrightarrow 1 \quad \text{for all } \delta>0. \tag{4.6}$$

Set $V_n(\beta) = \frac{1}{n} \log Z_n(\alpha, f(\alpha))$ and $X(\beta) = f(\alpha)/f(\alpha_0)$ if $\beta = (\alpha-\alpha_0)/\alpha_0$. Set also $M_n = N_n/n$ and $L'_n = \alpha_0 L_n$. We know that (see Lemma 1):

$$M_n \xrightarrow{P} 1, \qquad W'_n \xrightarrow{P} 1, \qquad L'_n/\ell_n \xrightarrow{P} 1, \tag{4.7}$$

and we may rewrite (4.4) as

$$V_n(\beta) = W'_n[1 - X(\beta)e^{-\beta L'_n}] + M_n[\log X(\beta) + \log(1+\beta) - \beta - \beta L'_n] + \beta U_n/\sqrt{n}. \tag{4.8}$$

Clearly, since $V_n(0)=0$ and $\hat{\alpha}_n$ is the argument of one of the maxima of $V_n(.)$, (4.6) will follow if we prove that for all $\delta>0$:

$$\lim_n P(\sup_{\beta:|\beta L'_n|>\delta} V_n(\beta) < 0) = 1 \tag{4.9}$$

We fix $\delta>0$. Note that $X(0)=1$ and X is continuous at 0, so for $C\in(0,1)$ fixed there is $C'>0$ with $|X(\beta)-1|\le C$ when $|\beta|\le C'$. Hence if $|\beta|\le C'$ and if we set $u_n = W'_n C + M_n[\log(1+C)(1+C') + C]$, $v_n = W'_n(1-C)$ and $w_n = M_n - U_n/L'_n\sqrt{n}$, we have on the set $\{W'_n>0\}$:

$$V_n(\beta) \le u_n + v_n(1-e^{-\beta L'_n}) - w_n \beta L'_n.$$

Now if $a>0$ and $1 < b/a < (e^{\delta}-e^{-\delta})/2\delta$, the function $x \to a(1-e^{-x})-bx$ reaches its maximum on the set $\{x:|x|\ge\delta\}$ at $x=\delta$, for which it equals $a(1-e^{-\delta})-b\delta$. Then on the set $A_n = \{W'_n>0\}\cap\{v_n>0\}\cap\{1<w_n/v_n<(e^{\delta}-e^{-\delta})/2\delta\}$,

$$V_n(\beta) \le u_n + v_n(1-e^{-\delta}) - w_n\delta \quad \text{if } |\beta|\le C',\ |\beta L'_n|\ge\delta.$$

But (4.7) yields $u_n \xrightarrow{P} u := C + C' + \log(1+C)(1+C')$, $v_n \xrightarrow{P} 1-C$ and $w_n \xrightarrow{P} 1$. We choose C,C' small enough so that $u + (1-C)(1-e^{-\delta}) - \delta < 0$ and that $1/(1-C) < (e^{\delta}-e^{-\delta})/2\delta$, so $P(A_n) \to 1$ and finally

$$\lim_n P(\sup_{\beta:|\beta L'_n|>\delta, |\beta|\le C'} V_n(\beta) < 0) = 1. \tag{4.10}$$

Next if $a,b>0$ we have $-ae^{-v}-bv \le b(-1 - \log a + \log b)$ for all $v\in\mathbb{R}$; applying this to $v = \beta L'_n$, $a = W'_n X(\beta)$, $b = M_n$, we deduce from (4.8) that on $\{W'_n>0\}$:

$$\begin{aligned} V_n(\beta) &\le W'_n + M_n[-1 - \log W'_n + \log(1-\beta) - \beta + \log M_n] + \beta U_n/\sqrt{n} \\ &\le M_n \log(1-\beta) + A_n\beta + B_n \end{aligned}$$

where $A_n := -M_n + U_n/\sqrt{n} \xrightarrow{P} -1$ and $B_n := W'_n - M_n(1 + \log W'_n + \log M_n) \xrightarrow{P} 0$; since $\sup_{\beta:|\beta|\ge C'}[\log(1-\beta) - \beta] < 0$ when $C'>0$, we deduce

$$\lim_n P(\sup_{\beta:|\beta|\ge C'} V_n(\beta) < 0) = 1. \tag{4.11}$$

Then (4.10) and (4.11) prove (4.9).

3) Now the proof of (4.5) goes along the traditional route. The derivative of $\alpha \to \log Z_n(\alpha,f(\alpha))$ is (see (4.4)):

$$U_n(\alpha) = nW'_n \frac{f(\alpha)}{f(\alpha_0)}[L_n - \frac{f'(\alpha)}{f(\alpha)}]e^{-(\alpha-\alpha_0)L_n} + N_n[\frac{f'(\alpha)}{f(\alpha)} + \frac{1}{\alpha}] - \overline{N}_n. \tag{4.12}$$

Let B_n be the set where the maximum of $Z_n(.,f(.))$ is reached inside A. On B_n we have $U_n(\check{\alpha}_n)=0$, hence Taylor's expansion yields

$$U_n(\alpha_0) + (\check{\alpha}_n-\alpha_0)U'_n(\beta_n) = 0, \quad \text{where } \beta_n \text{ is between } \alpha_0 \text{ and } \check{\alpha}_n. \tag{4.13}$$

By (4.12),

$$U'_n(\alpha) = -nW'_n \frac{f(\alpha)}{f(\alpha_0)}[(L_n - \frac{f'(\alpha)}{f(\alpha)})^2 + g(\alpha)]e^{-(\alpha-\alpha_0)L_n} + N_n[g(\alpha) - \alpha^{-2}]$$

where $g = (ff"-f'^2)/f^2$. We know that $\alpha_0 L_n/\ell_n \xrightarrow{P} 1$, and f, $f'f$ and g are locally bounded on A, hence by (4.7) and the continuity of f and the fact that $\alpha_0\in A$:

$$\lim_{\delta\to 0} \limsup_n P(\sup_{\alpha: |\ell_n(\alpha-\alpha_0)|\le\delta} |1 + \frac{U_n'(\alpha)}{nL_n^2}| > \eta) = 0,$$

for all $\eta>0$, and it follows by (4.6) that, with β_n as in (4.13):

$$U_n'(\beta_n)/n\ell_n^2 \xrightarrow{P} -1/\alpha_0^2. \tag{4.14}$$

Next (4.3) and (4.12) yield

$$\alpha_0 U_n(\alpha_0) = nW_n'(\alpha_0 L_n - c) + N_n(c+1) - \alpha_0 \overline{N}_n = [U_n + (\alpha_0 L_n - c)(W_n - V_n)]\sqrt{n},$$

and thus $U_n(\alpha_0)\omega_n = \overline{W}_n - \overline{V}_n + o_P(1)$. This, (4.14), (4.13) and the fact that $P(B_n) \to 1$ (which follows from (4.6)) readily give (4.5), and the proof is finished. ∎

REMARK 6: If $\ell_n \to \ell \in \mathbb{R}$ the statement (b) of Theorem 3 may be wrong, and it might even happen that the MLE $\check{\alpha}_n$ are not consistent.

However, if $f(\alpha) = \xi_0$ for all α, then whatever the limit ℓ is, $\check{\Gamma}_n$ is a central sequence for $\mathcal{E}_n$ and it converges in law to $\mathcal{N}(0,1/\sqrt{\varphi})$: the proof is very much along the same line as above. More generally, this statement is true when the function f is such that for all $L\in\mathbb{R}$,

$$\alpha \longrightarrow h_L(\alpha) := \frac{1}{\alpha} - \frac{1}{\alpha_0} + (\frac{f'(\alpha)}{f(\alpha)} - L)(1 - \frac{f(\alpha)}{f(\alpha_0)} e^{-(\alpha-\alpha_0)L})$$

vanishes only at $\alpha=\alpha_0$. ∎

REMARK 7: If one has the choice of the observation schemes, it is suitable to have ω_n go as fast as possible to 0, or equivalently $|\ell_n|$ go as fast as possible to $+\infty$. For Schemes 1 or 2 (resp. Schemes 1 or 3) it means that for n fixed, it is best to take y_n (resp. t_n) as small as possible. This is in contrast with the complete 2-dimensional model, for which the (best) rate of convergence of estimators of α is always $1/\sqrt{n}$, whatever schemes are chosen (Corollary 1). ∎

REMARK 8: It is obviously possible to undertake a similar study when α is a known function $g(\xi)$ of ξ, twice continuously differentiable on some open subset B of $(0,\infty)$. We consider two cases:

1) The derivative has $g'(\xi_0)\neq 0$: then in a neighbourhood of ξ_0 there is a bi-continuous one-to-one correspondance between $\alpha=g(\xi)$ and ξ, and the pre-

vious results for α-models are immediately transcribed in terms of ξ-models.

2) $g(\xi)=\alpha_0$ is constant on B: then for every fixed n, the model is an exponential family by (2.4). So UMVU estimators for certain functionals of ξ exist: take $(N_n/t_n)e^{\ell_n}$ as estimator for ξ in Scheme 1, $(T_n/n)e^{-\alpha_0 L_n}$ for $1/\xi$ in Schemes 2 or 3 (recall that α_0 is known here). By Lemma 1 these estimators are asymptotically normal with rate $1/\sqrt{n}$, and LAN property will hold for local models $\xi = \xi_0(1+H/\sqrt{n})$ at ξ_0 for all observation schemes. ■

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